Imperial/TP/050502
hep-th/0505207

Marginal Deformations of Field Theories with AdS_4 Duals

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Abstract

We generate new AdS_4 solutions of $D = 11$ supergravity starting from $AdS_4 \times X_7$ solutions where X_7 has $U(1)^3$ isometry. We consider examples where X_7 is weak G_2, Sasaki-Einstein or tri-Sasakian, corresponding to $d = 3$ SCFTs with $\mathcal{N} = 1, 2$ or 3 supersymmetry, respectively, and where the deformed solutions preserve $\mathcal{N} = 1, 2$ or 1 supersymmetry, respectively. For the special cases when X_7 is $M(3, 2)$, $Q(1, 1, 1)$ or $N(1, 1)_I$ we identify the exactly marginal deformation in the dual field theory. We also show that the volume of supersymmetric 5-cycles of $N(1, 1)_I$ agrees with the conformal dimension predicted by the baryons of the dual field theory.

1 Introduction

It is not uncommon for supersymmetric conformal field theories to have exactly marginal deformations that preserve the superconformal symmetry. If a conformal field theory with exactly marginal operators has an AdS dual [1] one should be able to construct whole families of AdS solutions. An early perturbative construction of such solutions was undertaken in [2] for certain deformations of $AdS_5 \times S^5$. In a recent work Lunin and Maldacena discovered a remarkably simple method for generating new AdS solutions (not necessarily supersymmetric) starting from an AdS solution with an internal space possessing certain isometries [3]. The new solutions describe a special class of exactly marginal deformations of the CFT dual to the original AdS solution.

One focus of [3] was type IIB AdS solutions with the internal space having a $U(1)^2$ isometry. Such solutions can also be viewed as solutions of the $D = 8$ supergravity theory obtained by compactifying type IIB on the two-torus. The duality group of this $D = 8$ supergravity is $SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$ (of which an $SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$ subgroup survives in string theory). The key observation of [3] was that starting from a given AdS solution there are certain $SL(2,\mathbb{R}) \subset SL(3,\mathbb{R})$ transformations that generate new *regular* AdS solutions. Furthermore, the corresponding exactly marginal deformations ("β-deformations") in the dual CFT were identified. The techniques were illustrated with several AdS_5 examples, including $AdS_5 \times S^5$, $AdS_5 \times T^{1,1}$ and $AdS_5 \times Y^{p,q}$ where $Y^{p,q}$ are the new Sasaki-Einstein spaces discovered in [4, 5]. The $Y^{p,q}$ spaces have recently been generalised in [6]. These new metrics have $U(1)^3$ isometry, of which a $U(1)^2$ preserves the Killing spinors, and we note that it is straightforward find the corresponding supersymmetric deformed solutions. Deformations of non-conformal geometries were also considered and a generalisation recently appeared in [7].

Lunin and Maldacena also generalised the technique to AdS solutions of $D = 11$ supergravity when the internal space has a $U(1)^3$ isometry. These solutions are also solutions of the $D = 8$ supergravity theory obtained by compactifying $D = 11$ supergravity on a three-torus, which is in fact the same $D = 8$ supergravity theory arising from the compactification of type IIB on a two-torus. In this case the key duality transformations lie in the explicit $SL(2,\mathbb{R})$ factor in $SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$. The technique was illustrated using the maximally supersymmetric $AdS_4 \times S^7$ solution.

Here we will study some additional AdS_4 examples in $D = 11$. The solutions that we use to generate the new solutions describing the deformations are all of the Freund-Rubin form: $AdS_4 \times X_7$. Such solutions arise as the near horizon limit of M2-branes lying at the singular apex of an eight-dimensional Ricci-flat cone whose base is X_7. We will focus on supersymmetric solutions in which case the cone must have special holonomy [8, 9, 10], or equivalently X_7 admits at least one Killing spinor. In order to find deformed solutions using the techniques of [3], we need examples where X_7 has $U(1)^3$ isometry. If we demand that the deformed geometry preserves some supersymmetry, which we mostly will, this T^3 action must preserve at least one Killing spinor on X_7.

We will find deformations for a number of different X_7, which we will review in

section 2. Specifically we will consider the tri-Sasakian metric on $N(1,1)$, which we denote by $N(1,1)_I$, the Sasaki-Einstein metrics on $Q(1,1,1)$ and $M(3,2)$ along with their co-homogeneity one generalisations found in [11]. We will also discuss the weak G_2 metrics on the squashed seven-sphere and on $N(k,l)$.

It is interesting that dual $d=3$ conformal field theories with $\mathcal{N}=2$ supersymmetry have been proposed for the $Q(1,1,1)$ and $M(3,2)$ solutions in [12] and with $\mathcal{N}=3$ supersymmetry for the tri-Sasaki metric $N(1,1)_I$ in [13] (see also [14]), and this will also be reviewed in section 2. Although these field theories are still not well understood, a spectrum of chiral operators consistent with the Kaluza-Klein spectrum can be obtained. For the $Q(1,1,1)$ and $M(3,2)$ cases the volumes of some supersymmetric 5-cycles have been shown to agree with the conformal dimensions of baryon operators. The latter calculation has not yet been performed for the $N(1,1)_I$ case so, as somewhat of an aside, we do this in an appendix finding results consistent with the field theory proposed in [13]. In addition, for all three of these examples we are able to identify the exactly marginal operators in the field theories corresponding to the deformed supergravity solutions that we construct.

The plan for the remainder of the paper is as follows. In section 3 we discuss the technique of constructing the deformed solutions and analyse how much of the supersymmetry is preserved. In section 4 we present the new deformed supergravity solutions. In section 5 we analyse the field theory duals. We conclude in section 6, which includes a discussion of some simple non-supersymmetric generalisations.

2 Review of some $AdS_4 \times X_7$ solutions and their duals

Let us begin by recalling some well known facts about $AdS_4 \times X_7$ solutions of $D=11$ supergravity with X_7 an Einstein manifold and $F_4 \propto vol_{AdS_4}$, where vol_{AdS_4} is the volume-form of AdS_4. We choose the metric on X_7 to be normalised so that $Ricci(X_7) = 6g(X_7)$, just as for a unit radius S^7. In order to preserve supersymmetry, the eight-dimensional cone over X_7 with metric

$$ds^2 = dr^2 + r^2 ds^2(X_7) \tag{2.1}$$

must have special holonomy. If the cone has $Spin(7)$ holonomy then X_7 is a weak G_2 manifold and the $D=11$ solution is dual to an $\mathcal{N}=1$ SCFT. If the cone has $SU(4)$ holonomy, i.e. it is a Calabi-Yau four-fold, then X_7 is a Sasaki-Einstein manifold and this is dual to an $\mathcal{N}=2$ SCFT. If the cone has $Sp(2)$ holonomy, i.e. is hyper-Kähler, then X_7 is tri-Sasakian and this is dual to an $\mathcal{N}=3$ SCFT. If the cone is flat we have the maximally supersymmetric $AdS_7 \times S^7$ solution dual to an $\mathcal{N}=8$ SCFT. This is summarised in the table below. Note that for SCFTs in three-dimensions with $\mathcal{N}$ supersymmetries the R-symmetry is $so(\mathcal{N})$.

X_7	Holonomy of $C(X_7)$	$\mathcal{N}$
Weak G_2	$Spin(7)$	1
Sasaki-Einstein	$SU(4)$ (Calabi–Yau)	2
tri-Sasaki	$Sp(2)$ (hyper-Kähler)	3
S^7	$\mathbf{1}$ (flat)	8

One class of examples that we study is when X_7 is homogeneous i.e. of the form G/H. The classification of all homogeneous Einstein manifolds in seven dimensions was carried out long ago in [15] (see [16] for a review) and discussed in an AdS/CFT setting in [17]. We have:

1. the Sasaki-Einstein space[1] $M(3,2)$ with $SU(3)\times SU(2)\times U(1)$ isometry [18, 19]. This is a regular Sasaki-Einstein manifold that is naturally a $U(1)$ bundle over $\mathbb{C}P^2 \times S^2$.

2. the Sasaki-Einstein space $Q(1,1,1)$ with $SU(2)^3 \times U(1)$ isometry [20]. This is also a regular Sasaki-Einstein manifold that is naturally a $U(1)$ bundle over $S^2 \times S^2 \times S^2$

3. the Aloff-Wallach spaces $N(k,l) = SU(3)/U(1)$ (see [21] for a recent discussion). For general k, l these admit two Einstein metrics, $N(k,l)_I$ and $N(k,l)_{II}$ but if $l = 0$ these two metric coincide [22]. The metric[2] $N(1,1)_I$ is tri-Sasakian [19] while its squashed version $N(1,1)_{II}$ as well as all other metrics on $N(k,l)$ are weak G_2 [19, 22]. Note that both metrics on $N(1,1)$ have $SU(3) \times SO(3)$ isometry, while the other metrics on $N(k,l)$ have $SU(3) \times U(1)$ isometry.

4. the squashed seven sphere, $SO(5) \times SO(3)/SO(3) \times SO(3)$. This manifold is weak G_2 and has $SO(5) \times SO(3)$ isometry.

Note that the weak G_2 manifold $SO(5)/SO(3)_{\max}$ does not have a T^3 action so that we cannot deform it using the method of [3]. The Sasaki-Einstein homogeneous space $V_{5,2} = SO(5) \times U(1)/SO(3) \times U(1)$ does have a T^3 action but it does not preserve any Killing-spinors and hence the deformed solution will necessarily break supersymmetry.

In addition we will also deform the $AdS_4 \times X_7$ solutions where X_7 is the new infinite class of Sasaki-Einstein spaces found in [11]. These spaces are co-homogeneity one generalisations of the $M(3,2)$ and $Q(1,1,1)$ spaces. In order to simplify the presentation, we will not explicitly discuss the generalisation discussed in [24, 25], nor the further generalisation discussed in [6], but it is straightforward to do so.

A summary of the supersymmetric solutions that we consider (including the S^7 example studied in [3]) and the amount of supersymmetry that we will show the

[1] $M(3,2)$ is also known as $M^{1,1,1}$.
[2] $N(1,1)_I$ is also known as $N^{0,1,0}$.

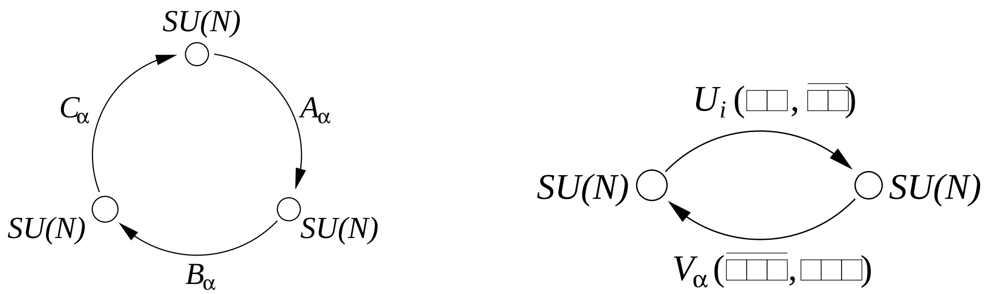

Figure 1: The quiver diagrams for the $Q(1,1,1)$ and $M(3,2)$ field theories.

deformed solutions preserve, $\mathcal{N}_\gamma$, is summarised below:

X_7	$\mathcal{N}$	$\mathcal{N}_\gamma$	Examples
S^7	8	2	
Tri-Sasaki	3	1	$N(1,1)_I$
Sasaki-Einstein	2	2	$Q(1,1,1), M(3,2)$ and generalisations.
Weak G_2	1	1	Squashed S^7, $N(k,l)_{I,II}$

(2.2)

2.1 Field theory

The above $AdS_4 \times X_7$ solutions are dual to the $d = 3$ supersymmetric CFTs arising on M2-branes sitting at the apex of the singular cone. Although our understanding of such field theories is still quite rudimentary, some concrete proposals have been made for the cases of $M(3,2)$ and $Q(1,1,1)$ in [12] and $N(1,1)_I$ in [13] (see also [14]) and some remarkable tests have been performed. The field theories arise from the IR limit of quiver gauge theories whose field content is summarized in Fig. 1 and Fig. 2.

The $\mathcal{N} = 2$ theory dual to $Q(1,1,1)$ theory has gauge group $SU(N)^3$. The isometries of $Q(1,1,1)$ give rise to $SU(2)^3 \times U(1)$ global symmetry, where the $U(1)$ factor is the R-symmetry. There are three chiral fields A, B and C which transform under the gauge group and $SU(2)^3$ in the representations summarised in the table below:

	$SU(N)_1$	$SU(N)_2$	$SU(N)_3$	$SU(2)_1$	$SU(2)_2$	$SU(2)_3$
A_α	$\boldsymbol{N}$	$\bar{\boldsymbol{N}}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$
B_α	$\mathbf{1}$	$\boldsymbol{N}$	$\bar{\boldsymbol{N}}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$
C_α	$\bar{\boldsymbol{N}}$	$\mathbf{1}$	$\boldsymbol{N}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$

(2.3)

They each have conformal dimension 1/3 (equal to their $U(1)_R$ charge). The chiral operators, which agree with the Kaluza-Klein spectrum, are given by $\mathrm{Tr}(ABC)^k$ symmetrised over all $SU(2)$ indices, i.e. in the $(\boldsymbol{k+1}, \boldsymbol{k+1}, \boldsymbol{k+1})$ rep, and have conformal dimension k. Note that one has to assume that other $SU(2)$ reps decouple in the IR (unlike the case of $T^{1,1}$ studied in [8] there is no superpotential suitable for this task). The second betti number of $Q(1,1,1)$, $b_2(Q(1,1,1))$, is two so the AdS/CFT correspondence predicts that the field theory has an extra $U(1)^2$ baryonic global symmetry. The manifold $Q(1,1,1)$ has three supersymmetric 5-cycles

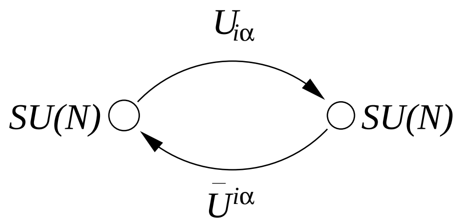

Figure 2: The quiver diagram for the $N(1,1)_I$ field theory.

(corresponding to divisors in the eight-dimensional cone) and wrapping five-branes on these cycles predicts baryons with conformal dimension $N/3$ in accord with the operators $\det A$, $\det B$ and $\det C$. Furthermore, for the global symmetry groups, the representations of the five-brane states and the operators have also been shown to agree.

Let us next describe the $\mathcal{N}=2$ theory dual to $M(3,2)$, which has gauge group $SU(N)^2$. The isometries of $M(3,2)$ give rise to $SU(3)\times SU(2)\times U(1)$ global symmetry, where the $U(1)$ factor is again the R-symmetry. There are two chiral fields U, and V which transform in the following representations:

	$SU(N)_1$	$SU(N)_2$	$SU(3)$	$SU(2)$
U_i	$Sym^2(C^N)$	$Sym^2(C^{N*})$	$\mathbf{3}$	$\mathbf{1}$
V_α	$Sym^3(C^{N*})$	$Sym^3(C^N)$	$\mathbf{1}$	$\mathbf{2}$

(2.4)

The field U has conformal dimension 4/9 while V has dimension 1/3. The chiral operators given by $\mathrm{Tr}(U^3V^2)^k$, totally symmetrised over the $SU(3)\times SU(2)$ indices, have dimension $2k$ and these agree with the Kaluza-Klein spectrum. Again one has to assume that other $SU(3)\times SU(2)$ reps decouple in the IR. Since $b_2(M(3,2))=1$ the field theory is predicted to have a $U(1)$ baryonic symmetry. The manifold $M(3,2)$ has two supersymmetric 5-cycles and wrapping five-branes on these cycles predicts baryons with conformal dimensions $4N/9$ and $N/3$ in accord with the operators $\det U$ and $\det V$, respectively. In addition, for the global symmetry groups, the representations of the five-brane states and the operators have also been shown to agree.

The $\mathcal{N}=3$ field theory dual to $N(1,1)_I$ has gauge group $SU(N)^2$. The isometries of $N(1,1)_I$ are (locally) $SU(3)\times SU(2)$ where the $SU(2)$ factor is the R-symmetry. The theory contains three hypermultiplets transforming as a triplet under $SU(3)$. In terms of $\mathcal{N}=2$ superfields, these can be written as two sets of chiral fields u^i and v_i, $i=1,2,3$ transforming in the $\mathbf{3}$ and the $\bar{\mathbf{3}}$ of $SU(3)$ respectively. The $SU(2)_R$ action is best described by grouping these fields into $U^i_\alpha=(u^i,-\bar{v}^i)$ and $V_{i\alpha}=(v_i,\bar{u}_i)=-\epsilon_{\alpha\beta}\bar{U}^\beta_i$, each transforming as a doublet of $SU(2)_R$. We can summarise this is as:

	$SU(3)$	$SU(2)$	$SU(N)_1$	$SU(N)_2$
U^i_α	$\mathbf{3}$	$\mathbf{2}$	$\boldsymbol{N}$	$\bar{\boldsymbol{N}}$
$V_{i\alpha}$	$\bar{\mathbf{3}}$	$\mathbf{2}$	$\bar{\boldsymbol{N}}$	$\boldsymbol{N}$

(2.5)

Unlike in the other two field theories discussed above, it was proposed in [13] (using an $\mathcal{N}=2$ superfield description) that one can write down a suitable superpotential

which does not vanish after taking the trace over the gauge indices. The resulting equations of motion imply $u^i v_j u^j = -u^j v_j u^i$ and $v_i u^j v_j = -v_j u^j v_i$. It would be interesting to formulate this directly in an $\mathcal{N} = 3$ language.

The chiral operators with dimension k that agree with the Kaluza-Klein spectrum can be written as

$$\mathrm{Tr}\ U^{(i_1} V_{(j_1} \dots U^{i_k)} V_{j_k)} \tag{2.6}$$

where it must be assumed that the operators are symmetrised over $SU(3)$ indices and $SU(2)$ indices (not shown). We also demand that the operators are actually traceless and hence in the irreducible representation $(\boldsymbol{k}, \boldsymbol{k})$ of $SU(3)$ corresponding to the Young tableau:

$$\underbrace{\begin{array}[t]{|c|c|c|}\hline & \cdots & \\ \hline & \cdots & \\ \hline \end{array}}_{k}\underbrace{\begin{array}[t]{|c|c|c|}\hline & \cdots & \\ \hline \end{array}\vphantom{\begin{array}{c} x \\ x \end{array}}}_{k} \tag{2.7}$$

Other reps are assumed to decouple in the IR. It is interesting to note that the superpotential ensures that some of these operators are traceless. The theory has a single $U(1)$ baryonic symmetry since $b_2(N(1,1)_I) = 1$. We show in appendix A that $N(1,1)_I$ has supersymmetric 5-cycles whose volumes give rise to baryons with dimension $N/2$, in agreement with the conformal dimension of the operator $\det U$.

Although we shall not be considering it much in the following we note that a field theory dual to the $V_{5,2}$ case is discussed in [26]. As far as we are aware no proposal has been made for the case of the squashed seven sphere nor for other $N(k,l)_{I,II}$ cases. The field theories dual to the new Sasaki-Einstein spaces presented in [11] have not yet been worked out, but it seems plausible that once the toric description of the corresponding Calabi-Yau fourfold cones is found, generalising the results of [27], a proposal can be found by following the arguments of [12].

3 Deformation of Supergravity Solutions

In this section, after introducing our conventions for $D = 11$ supergravity, we will present the generating procedure of [3] in a form convenient for our purposes. We also analyse how much supersymmetry can be preserved in the deformed solutions.

3.1 Conventions

Our conventions for the bosonic fields of $11d$ supergravity are as in [28]. In particular the Lagrangian is

$$2\kappa^2 \mathcal{L} = R * 1 - \frac{1}{2} F_4 \wedge * F_4 - \frac{1}{6} C_3 \wedge F_4 \wedge F_4 \,. \tag{3.1}$$

The $11d$ Planck length is defined by $2\kappa^2 = (2\pi)^8 l_{11}^9$. In these conventions, the dimensionless integer valued M2- and M5-brane charges are expressed in terms of the flux

of F_4 as

$$N_{M2} = \frac{1}{(2\pi l_{11})^6} \int_{C_7} *F_4\,, \qquad N_{M5} = \frac{1}{(2\pi l_{11})^3} \int_{C_4} F_4\,, \tag{3.2}$$

for some 7-cycle C_7 or 4-cycle C_4 surrounding the branes.

The main subject of this paper is deformations of backgrounds of the type $AdS_4 \times X_7$ which arise as the near horizon limit of branes sitting at the apex of the eight-dimensional cone with base X_7. The undeformed solution takes the form

$$ds^2 = R_{X_7}^2 \left(\frac{1}{4} ds^2_{AdS_4} + ds^2_{X_7} \right), \quad F_{(4)} = \frac{3}{8} R_{X_7}^3 vol_{AdS_4}\,, \tag{3.3}$$

provided that we normalize $ds^2_{X_7}$ such that $R_{\mu\nu}(X_7) = 6 g_{\mu\nu}(X_7)$, just as for a unit radius S^7. The radius R_{X_7} is determined by the quantisation condition (3.2),

$$6 R_{X_7}^6 \mathrm{Vol}(X_7) = (2\pi l_{11})^6 N\,, \tag{3.4}$$

with $\mathrm{Vol}(X_7)$ being the volume of X_7. For the special case of a unit radius S^7 we have $\mathrm{Vol}(S^7) = \pi^4/3$ and hence $R_{S^7} = (32\pi^2 N)^{1/6} l_{11}$. Finally, we note that the choice of orientation for AdS_4 (or equivalently X_7) implicit in the expression for the four-form is chosen so that the solution is supersymmetric: recall that if, for example, we flip the sign of F_4 then we obtain a solution to the equations of motion that does not preserve any supersymmetry (apart from the special case when X_7 is the round seven-sphere) [29].

3.2 Transformation Rule

The full symmetry group of the $11d$ supergravity compactified on a three-torus is $SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$. We will consider supergravity solutions with $U(1)^3$ isometry and deform them by an element of $SL(2,\mathbb{R})$. The action of the symmetry group can be deduced from the following Kaluza-Klein ansatz:[3]

$$\begin{aligned} ds^2_{11} &= \Delta^{1/3} M_{ab} D\varphi^a D\varphi^b + \Delta^{-1/6} g_{\mu\nu} dx^\mu dx^\nu\,, &(3.5)\\ C_3 &= C_{(0)} D\varphi^1 D\varphi^2 D\varphi^3 + \frac{1}{2} C_{(1)ab} D\varphi^a D\varphi^b + C_{(2)a} D\varphi^a + C_{(3)}\,, &(3.6) \end{aligned}$$

where $D\varphi^a = d\varphi^a + A^a_\mu dx^\mu$, $\det(M_{ab}) = 1$ and the products of forms in the last line are wedge products. The $SL(3,\mathbb{R})$ part of the symmetry is manifest. The transformation rule under the $SL(2,\mathbb{R})$ part is less trivial and can be found, for example, in [30]. The $8d$ Einstein metric $g_{\mu\nu}$ and $C_{(2)a}$ are invariant. The τ-parameter is defined by $\tau \equiv -C_{(0)} + i\sqrt{G_{T^3}} = -C_{(0)} + i\Delta^{1/2}$ and it transforms in the usual way

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})\ ; \quad \tau \to \frac{a\tau + b}{c\tau + d}\,. \tag{3.7}$$

[3]To avoid clutter, we will often suppress the overall radius R in the intermediate steps.

The $8d$ vectors A^a_μ and $C_{(1)ab\mu}$ form a doublet in the same way as the NSNS and RR two-form fields do in the $10d$ IIB supergravity,

$$B^a = \begin{pmatrix} A^a \\ -\frac{1}{2}\epsilon^{abc}C_{(1)bc} \end{pmatrix}, \qquad B^a \to \Lambda^{-T} B^a \,. \tag{3.8}$$

Finally, and somewhat unexpectedly, the field strength corresponding to $C_{(3)}$ also forms a doublet with its magnetic dual:

$$H = \begin{pmatrix} F_{(4)} \\ \Delta^{1/2} *_8 F_{(4)} + C_{(0)}F_{(4)} \end{pmatrix}, \qquad H \to \Lambda^{-T} H \,, \tag{3.9}$$

where the Hodge dual is taken with respect to the $8d$ metric.

Now, we are ready to apply the transformation rules to deform the background (3.3) using a $U(1)^3$ action on X_7. Since the 4-form field strength lies entirely in AdS_4, we have $C_{(0)} = C_{(1)} = C_{(2)} = 0$ in the Kaluza-Klein ansatz (3.6), and the only non-vanishing field strength is $F_{(4)} = dC_{(3)}$. As shown in [3] the deformed metric is regular only for the $SL(2,\mathbb{R})$ elements of the form

$$\Lambda = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} . \tag{3.10}$$

We shall only consider these transformations and call the corresponding transformed solutions "γ-deformed solutions".

It follows that once we express the metric in the KK form (3.5), $g_{\mu\nu}$, A^a are invariant, while $C_{(1)ab}$, $C_{(2)a}$ remain zero. There are only two non-trivial transformation rules. The first is $\tau' = \tau/(1+\gamma\tau)$ which gives

$$\Delta' = G^2\Delta \,, \qquad C'_{(0)} = -\gamma G \Delta \,, \tag{3.11}$$

where

$$G \equiv \frac{1}{(1+\gamma^2\Delta)} \,. \tag{3.12}$$

Combining this with the other transformation concerning $F_{(4)}$, we find

$$F'_4 = F_4 - \gamma\Delta^{1/2} *_8 F_4 - \gamma d\left(G\Delta D\varphi^1 D\varphi^2 D\varphi^3\right) . \tag{3.13}$$

To summarise, to obtain the γ-deformed solution we first write the metric on X_7 in the form

$$ds^2_{X_7} = ds^2_{T^3} + ds^2_4 \,, \qquad ds^2_{T^3} := \Delta^{1/3} M_{ab} D\varphi^a D\varphi^b \,, \tag{3.14}$$

with $D\varphi^a \equiv d\varphi^a + A^a$ and $\det(M_{ab}) = 1$. The quantities Δ, M_{ab} and A^a will depend on the particular X_7 being considered. The solution obtained by the $SL(2,\mathbb{R})$ transformation (3.10) is simply

$$ds^2_{11} = G^{-1/3}\left(\frac{1}{4} ds^2_{AdS_4} + ds^2_4 + G\, ds^2_{T^3}\right) , \tag{3.15}$$

$$F_4 = \frac{3}{8} vol_{AdS_4} - 6\,\gamma\,\Delta^{1/2}\, vol_{ds_4} - \gamma\, d(G\Delta\ D\varphi_1 \wedge D\varphi_2 \wedge D\varphi_3) \,, \tag{3.16}$$

where vol_{ds_4} is the volume form[4] of ds_4^2 and $G = (1+\gamma^2\Delta)^{-1}$. The dependence on the radius R_{X_7} can be reinstated by multiplying ds_{11}^2 by $R_{X_7}^2$, F_4 by $R_{X_7}^3$ and replacing γ by $\hat{\gamma} = (R_{X_7}/l_{11})^3\gamma$.

It is interesting to consider the term $6\gamma\Delta^{1/2}vol_{ds_4}$ in the four-form field strength. In the case that the base-space M_4, with local metric ds_4^2, is actually a manifold, this term corresponds to γN M5-branes wrapped on the T^3, where N is, of course, the number of membranes creating the background. To see this, we first re-instate R_{X_7} dependence which brings this piece of the 4-form, which we denote by f_4, to the form

$$f_4 = 6\frac{R_{X_7}^6}{l_{11}^3}\gamma\Delta^{1/2}vol_{ds_4}\,. \tag{3.17}$$

Using (3.2) and (3.4), we then have

$$N_{M5} = \frac{1}{(2\pi l_{11})^3}\int_{M_4} f_4 = \frac{\gamma N}{\mathrm{Vol}(X_7)}\int_{M_4}(2\pi)^3\Delta^{1/2}vol_{ds_4} = \gamma N. \tag{3.18}$$

We have used the fact that if the co-ordinates φ^a have period 2π, then $(2\pi)^3\Delta^{1/2}$ is the volume of the T^3 fibre at any point on the base M_4 and the second integral then equals $\mathrm{Vol}(X_7)$. These M5-branes are sitting at a constant radial position in AdS_4 and they wrap the T^3. The latter is a contractible cycle so this configuration can be interpreted as membranes expanded to a form a fuzzy T^3 via the Myer's effect.

We note that choosing the coordinates φ^a to have period 2π ensures that the $SL(2,\mathbb{Z})$ transformations with integer entries in (3.7) is the exact $SL(2,\mathbb{Z})$ symmetry of the full M-theory. This is consistent with N_{M5} in (3.18) being integer valued when γ is.

3.3 Preservation of Supersymmetry

We focus on deformations which preserve some supersymmetry. Thus we demand that the T^3 action should commute with at least one supersymmetry. i.e. the Lie derivative with respect to the three $U(1)$'s of at least one Killing spinor should vanish. If this is not the case, then the deformed solution will not preserve any supersymmetry and we briefly return to this possibility at the end of the paper.

We normalise our seven-dimensional metrics so that $Ricci(X_7) = 6g(X_7)$. Choosing the seven-dimensional hermitian gamma-matrices to be imaginary and satisfy $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$, the real Killing spinors ψ satisfy

$$\nabla_a\psi = \pm\frac{i}{2}\gamma_a\psi. \tag{3.19}$$

The maximally supersymmetric case of S^7 has eight Killing spinors for each sign, but for all other cases the Killing spinors have the same sign (which is fixed by the orientation of X_7). Take it to be the plus sign for definiteness. If X_7 admits a Killing

[4]The orientation for ds_4 is chosen so that the positive orientation for X_7, canonically fixed by it's Killing spinors, is equal to $vol_{ds_4}d\varphi^1 d\varphi^2 d\varphi^3$.

vector k^a then we can define the spinorial Lie-derivative. Acting on a Killing spinor we have

$$\begin{aligned} \mathcal{L}_k\psi &\equiv [k^a\nabla_a + \frac{1}{4}\nabla_a k_b\gamma^{ab}]\psi \\ &= [\frac{i}{2}k^a\gamma_a + \frac{1}{4}\nabla_a k_b\gamma^{ab}]\psi\,. \end{aligned} \tag{3.20}$$

It is well known that the spinorial Lie-derivative preserves the space of Killing spinors (see e.g. [31]).

Consider now X_7 to be a weak G_2 manifold with precisely one Killing spinor ψ. These give solutions dual to $\mathcal{N}=1$ CFTs which have no R-symmetry. If k is a Killing vector on X_7 then $\mathcal{L}_k\psi$ is also a Killing spinor and hence proportional to ψ: $\mathcal{L}_k\psi = \alpha\psi$ with α real. However, multiplying by ψ^T and using the fact that γ^a and γ^{ab} are both anti-symmetric we conclude that $\alpha = 0$. Thus the single Killing spinor must be invariant. Thus, quite generally, any weak G_2 manifold with a T^3 action can be deformed and the deformed solution will preserve one supersymmetry. We will discuss some explicit examples below.

Next consider X_7 to be Sasaki-Einstein with two Killing spinors. These give solutions dual to $\mathcal{N}=2$ CFTs which have a $U(1)$ R-symmetry (or, in fact, a non-compact $\mathbb{R}$ R-symmetry), and this corresponds to the fact that the Sasaki-Einstein manifold has a canonical Killing-vector, sometimes called the Reeb vector. Identifying $U(1)$ with $SO(2)$, the two Killing spinors transform as a **2** (see e.g. [31]). Thus if we are seeking deformed solutions that preserve supersymmetry we cannot use the Reeb vector as part of the T^3 action. It is not difficult to show, using (3.20), that if a T^3 action leaves one of the two Killing spinors invariant, it will in fact leave both invariant. To see this, we can take ψ_1 and ψ_2 be two orthogonal Killing spinors with $\mathcal{L}_k\psi_1 = 0$ and $\mathcal{L}_k\psi_2 = \alpha\psi_1 + \beta\psi_2$. If we now multiply the second equation by ψ_2^T we deduce that $\beta = 0$. On the other hand, if we multiply by ψ_1^T and use the first equation we then deduce that $\alpha = 0$. Solutions deformed using such a T^3 action will thus preserve two supersymmetries.

Finally, we consider X_7 to be tri-Sasaki with three Killing spinors. These give solutions dual to $\mathcal{N}=3$ CFTs which have an $SO(3)$ R-symmetry, and this corresponds to the fact that the Sasaki-Einstein manifold has three canonical Killing-vectors generating an $SO(3)$. The three Killing spinors transform as a **3** of $SO(3)$ (see e.g. [31]). The tri-Sasaki-Einstein manifolds and T^3 actions that we consider lead to deformations that preserve one supersymmetry. We first choose a $U(1)$ sub-group of the $SO(3)$ R-symmetry. Two of the Killing spinors are charged with respect to this $U(1)$ while the third is neutral. Thus, a suitable T^3 action is obtained by supplementing this $U(1)$ action with a T^2 action that also leaves this single Killing spinor invariant.

We now turn to some detailed examples.

4 Examples

In this section we construct the γ-deformed solutions for a number of different examples. As explained above, for each example, we just need to compute the quantities

$(\Delta, A^i, M_{ab}, ds_4^2)$ defined in (3.14) for a specified T^3 action. The metric and the four-form of the γ-deformed solution is then given by (3.15) and (3.16).

4.1 Sasaki-Einstein

Let us begin by recalling a few facts about Sasaki-Einstein (SE) manifolds. Locally, any seven-dimensional SE metric can be put in the form

$$ds^2 = (d\psi' + \sigma)^2 + ds_6^2(M_6)\,, \tag{4.1}$$

where ds_6^2 is a local metric on a Kähler-Einstein base-space M_6. If J_6 and Ω_6 are the Kähler-form and $(3,0)$-form on M_6 respectively, we have $d\sigma = 2J_6$ and $d\Omega_6 = 4i\sigma\wedge\Omega_6$. The vector $\partial_{\psi'}$ is the Reeb vector which is Killing. If the Reeb vector has closed orbits and the $U(1)$ action is free, we have a regular SE manifold. In this case M_6 is a manifold. If the $U(1)$ action has finite isotropy groups then we have a quasi-regular SE and M_6 is an orbifold. If the orbits of the Reeb vector do not close we have an irregular SE, and M_6 is not globally defined in any sense. The field theories dual to regular and quasi-regular SE manifolds have a $U(1)$ R-symmetry, while those dual to irregular SE manifolds have a non-compact $\mathbb{R}$ R-symmetry.

We seek SE manifolds with a T^3 action commuting with the two Killing spinors. The two Killing spinors can be constructed locally from the gauge-covariantly constant spinors on the Kähler-Einstein base space (see e.g. [33]). From this description we deduce that the T^3 action must be independent of the Reeb vector and leave the gauge-covariantly constant spinors on the base invariant. A convenient way to check the latter is to check that the T^3 action leaves J_6 and Ω_6 invariant.

We will consider the homogeneous examples $M(3,2)$ and $Q(1,1,1)$, both of which are regular SE manifolds. $M(3,2)$ is a $U(1)$ bundle over $\mathbb{C}P^2 \times S^2$ with winding numbers 3 and 2 over $\mathbb{C}P^2$ and S^2, respectively. $Q(1,1,1)$ is a $U(1)$ bundle over $S^2 \times S^2 \times S^2$ with unit winding over each S^2. These metrics are incorporated in the cohomegeneity one manifolds found in [11] and we will consider the entire family. This family includes both quasi-regular and irregular SE manifolds.

We will follow the description of the relevant X_7 given in Ref. [11], where a general method of building up a SE manifold Y_{2n+3} from a Kähler-Einstein manifold B_{2n} is explained. For the case at hand, $n = 2$, the metric is given by [11]

$$ds^2_{X_7} = U(\rho)^{-1}d\rho^2 + \rho^2 ds^2_{B_4} + q(\rho)\left(d\psi + j_1\right)^2 + \omega(\rho)\left(d\alpha + f(\rho)(d\psi + j_1)\right)^2\,, \tag{4.2}$$

where $ds^2_{B_4}$ is the metric of the $4d$ base and the one-form j_1 satisfies $dj_1 = 2J$, with J being the Kähler form of B_4. The normalisation condition $R_{\mu\nu}(X_7) = 6g_{\mu\nu}(X_7)$ is equivalent to $R_{\mu\nu}(B_4) = 2g_{\mu\nu}(B_4)$. In this convention, the functions of ρ appearing in (4.2) are given by

$$\begin{aligned} U(\rho) &= \frac{1}{3} - \rho^2 + \frac{\kappa}{768\rho^6}\,, \qquad \omega(\rho) = \rho^2 U(\rho) + (\rho^2 - 1/4)^2\,, \\ f(\rho) &= \frac{\rho^2\left(U(\rho) + \rho^2 - 1/4\right)}{\omega(\rho)}\,, \qquad q(\rho) = \frac{\rho^2 U(\rho)}{16\,\omega(\rho)}\,. \end{aligned} \tag{4.3}$$

See [11] for more discussion on the possible values that κ can take and the range of the co-ordinate ρ. The two possible examples of B_4 with explicitly known metric are $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2$. We will discuss both of them.

The isometry group of X_7 always includes a $U(1)^4$ factor. Two of the $U(1)$'s come from the base B_4. With a suitable choice of coordinates, the two $U(1)$'s will be generated by $(\partial_{\phi_1}, \partial_{\phi_1})$. The other two $U(1)$'s are $(\partial_\alpha, \partial_\psi)$ appearing in (4.2). As explained in [11], the combination $V = \partial_\psi - \partial_\alpha$ corresponds to the Reeb vector or equivalently the R-symmetry of the superconformal algebra. We will perform the $SL(2,\mathbb{R})$ transformation on the T_3 generated by $(\partial_{\phi_1}, \partial_{\phi_2}, \partial_\alpha)$. In each case we have checked that the action does indeed leave the two Killing spinors invariant. We also note, that in general, the coordinate α does not have period 2π.

4.1.1 $S^2 \times S^2$ base

If we choose the four dimensional base $B_4 = S^2 \times S^2$, then the metric on X_7 is

$$\begin{aligned} ds_7^2 &= U^{-1} d\rho^2 + \frac{\rho^2}{2}\left(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2\right) \\ &\quad + q\left(d\psi + j_1\right)^2 + \omega\left(d\alpha + f(d\psi + j_1)\right)^2 , \end{aligned} \tag{4.4}$$

where the 1-form $j_1 = -\cos\theta_1 d\phi_1 - \cos\theta_2 d\phi_2$ is such that $2J_2 = dj_1$, J_2 being the Kähler form B_4. We have chosen the radius squared of the two S^2s to be $1/2$ to give the correct normalisation.

As explained above, to obtain the γ-deformed solution corresponding to the T^3 generated by $(\partial_{\phi_1}, \partial_{\phi_2}, \partial_\alpha)$, it suffices to give:

$$\begin{aligned} \Delta &= \frac{\rho^2\omega}{4}\left[q(1 - c_{2\theta_1}c_{2\theta_2}) + \rho^2 s_{\theta_1}^2 s_{\theta_2}^2\right] , \\ A^1 &= -\frac{q\rho^2\omega c_{\theta_1} s_{\theta_2}^2}{2\Delta} d\psi , \qquad A^2 = -\frac{q\rho^2\omega c_{\theta_2} s_{\theta_1}^2}{2\Delta} d\psi , \qquad A^3 = \frac{f\rho^4\omega s_{\theta_1}^2 s_{\theta_2}^2}{4\Delta} d\psi , \\ M_{ab} &= \Delta^{-1/3} \begin{pmatrix} (q+\omega f^2)c_{\theta_1}^2 + \frac{1}{2}\rho^2 s_{\theta_1}^2 & (q+\omega f^2)c_{\theta_1}c_{\theta_2} & -f\omega c_{\theta_1} \\ \cdot & (q+\omega f^2)c_{\theta_2}^2 + \frac{1}{2}\rho^2 s_{\theta_2}^2 & -f\omega c_{\theta_2} \\ \cdot & \cdot & \omega \end{pmatrix} , \\ ds_4^2 &= U^{-1} d\rho^2 + \frac{\rho^2}{2}\left(d\theta_1^2 + d\theta_2^2\right) + \frac{q\omega\rho^4 s_{\theta_1}^2 s_{\theta_2}^2}{4\Delta} d\psi^2 , \end{aligned}$$

where we have introduced the notation $s_\theta = \sin\theta$, $c_\theta \equiv \cos\theta$, etc. Note also that we have not entered all elements of the symmetric matrix M. The $D = 11$ solution is then given by (3.15) and (3.16).

4.1.2 $\mathbb{C}P^2$ base

We use the following metric for $\mathbb{C}P^2$:

$$\frac{ds^2_{\mathbb{C}P^2}}{3} = \sum_{i=1}^{3} d\mu_i^2 + \mu_1^2 d\phi_1^2 + \mu_2^2 d\phi_2^2 - (\mu_1^2 d\phi_1 + \mu_2^2 d\phi_2)^2 , \qquad \sum_{i=1}^{3} \mu_i^2 = 1 . \tag{4.5}$$

We have chosen the $\mathbb{C}P^2$ to have radius squared 3 to give the correct normalisation. With this choice of the radius, the Kähler form and potential read

$$\begin{aligned} J_2 &= 3\left(\mu_1 d\mu_1 d\phi_1 + \mu_2 d\mu_2 d\phi_2\right) , \\ j_1 &= 3\left(\mu_1^2 d\phi_1 + \mu_2^2 d\phi_2\right) , \qquad 2J_2 = dj_1 . \end{aligned} \tag{4.6}$$

These coordinates are inherited from the well known reduction $\mathbb{C}^3 \to S^5 \to \mathbb{C}P^2$,

$$w_1 = \mu_1 e^{i(\psi+\phi_1)}, \ \ w_2 = \mu_2 e^{i(\psi+\phi_2)}, \ \ w_3 = \mu_3 e^{i\psi} , \tag{4.7}$$

and are related to the standard inhomogeneous coordinates with the Fubini-Study Kähler potential K in the obvious way,

$$K = 3\log(1 + |z_1|^2 + |z_2|^2) , \qquad z_i = w_i/w_3 , \qquad i = 1, 2 . \tag{4.8}$$

The 7-manifold is

$$ds_7^2 = U^{-1} d\rho^2 + \rho^2 ds^2_{\mathbb{C}P^2} + q\left(d\psi - j_1\right)^2 + \omega\left(d\alpha + f(d\psi - j_1)\right)^2 . \tag{4.9}$$

The computation of the relevant data for the γ-deformed solution using the T_3 generated by $(\partial_{\phi_1}, \partial_{\phi_2}, \partial_\alpha)$ give:

$$\begin{aligned} \Delta &= 9\rho^2\omega\mu_1^2\mu_2^2\left[3q(\mu_1^2+\mu_2^2) + \rho^2\mu_3^2)\right] , \\ A^1 &= \frac{9q\rho^2\omega\mu_1^2\mu_2^2}{\Delta} d\psi , \qquad A^2 = \frac{9q\rho^2\omega\mu_1^2\mu_2^2}{\Delta} d\psi , \qquad A^3 = \frac{9f\rho^4\omega\mu_1^2\mu_2^2\mu_3^2}{\Delta} d\psi , \\ \Delta^{1/3} M_{ab} &= \begin{pmatrix} 9(q+\omega f^2)\mu_1^4 + 3\rho^2\mu_1^2(1-\mu_1^2) & 3(3q+3\omega f^2-\rho^2)\mu_1^2\mu_2^2 & 3f\omega\mu_1^2 \\ \cdot & 9(q+\omega f^2)\mu_2^4 + 3\rho^2\mu_2^2(1-\mu_2^2) & 3f\omega\mu_2^2 \\ \cdot & \cdot & \omega \end{pmatrix} \\ ds_4^2 &= U^{-1}d\rho^2 + 3\rho^2\sum_i d\mu_i^2 + \frac{9qw\rho^4\mu_1^2\mu_2^2\mu_3^2}{\Delta} d\psi^2 . \end{aligned}$$

4.2 Tri-Sasaki

We next consider the tri-Sasaki metric on $N(1,1)$. Topologically $N(1,1)$ is the coset space $SU(3)/U(1)$ where the $U(1)$ is embedded in the $\lambda_8 = \frac{1}{\sqrt{3}}\mathrm{diag}(1,1,-2)$ direction. As a bundle, one way it can be viewed is a single $SO(3)$ instanton on $\mathbb{C}P^2$ (the analogue of viewing S^7 as an instanton on S^4). It admits two Einstein metrics: $N(1,1)_I$ is tri-Sasaki, while $N(1,1)_{II}$ is weak G_2. In coordinates adapted to the $SO(3)$ fibration, both metrics can be written in the form [22],

$$\begin{aligned} 2ds^2 &= d\mu^2 + \frac{1}{4}\sin^2\mu(\sigma_1^2+\sigma_2^2) + \frac{1}{4}\sin^2\mu\cos^2\mu\,\sigma_3^2 \\ &\quad + \lambda^2\left\{(\Sigma_1 - \cos\mu\sigma_1)^2 + (\Sigma_2 - \cos\mu\sigma_2)^2 + (\Sigma_3 - \tfrac{1}{2}(1+\cos^2\mu)\sigma_3)^2\right\} , \end{aligned} \tag{4.10}$$

where Σ_i are right invariant one-forms on $SO(3)$, and σ_i are right invariant one-forms on $SU(2)$. This form of the metric is discussed in appendix A. The parameter

$\lambda^2 = 1/2$ for $N(1,1)_I$ and $\lambda^2 = 1/10$ for $N(1,1)_{II}$. The isometry group for both $N(1,1)_I$ and $N(1,1)_{II}$ is $SU(3) \times SO(3)$.

For the purpose of deforming the $N(1,1)_I$ metric and also comparing to the corresponding deformation of the field theory, it is convenient to use an alternative set of coordinates, naturally adapted to the construction of the metric as a hyper-Kähler quotient (see e.g. [13]). The relation between the two set of coordinates is explained in appendix B. The quotient construction starts with the flat metric for $\mathbb{C}^6$ parametrised by (u^i, v_i). There is an action of $SU(3)$ with u^i transforming as in the $\mathbf{3}$ representation and v_i in the $\bar{\mathbf{3}}$ representation. There is also an $SU(2)$ action where $(u^i, -\bar{v}^i)$ transforms as a doublet. These actions descend to $N(1,1)_I$ and generate the $SU(3) \times SO(3)$ isometry group. Note that the representations are the same as those of the fields in the dual field theory described in section 2.1 and hence we use the same notation. The hyper-Kähler quotient is built from the $U(1)$ action $(u^i, v_i) \to (e^{i\theta} u^i, e^{-i\theta} v_i)$. Viewing $\mathbb{C}^6$ as three copies of the quaternions, this action preserves the corresponding triplet of complex structures. The resulting moment maps give the constraints

$$|u^i|^2 - |v_i|^2 = 0 = u^i v_i. \tag{4.11}$$

If, in addition, we mod out by $(u^i, v_i) \sim (e^{i\theta} u^i, e^{-i\theta} v_i)$, the resulting space is the eight-dimensional cone over $N(1,1)_I$. Fixing the radial direction by taking $|u^i|^2 = |v_i|^2 = 1$ gives the metric on $N(1,1)_I$. As discussed in appendix B, before modding out by the $U(1)$ action the constrained (u^i, v_i) satisfying $|u^i|^2 = |v_i|^2 = 1$ and $u^i v_i = 0$ actually parametrise $SU(3)$. In this way one can see that $N(1,1)_I \simeq SU(3)/U(1)$.

To deform $N(1,1)_I$ we need to pick out three $U(1)$ Killing directions: a $U(1)^2$ from the $SU(3)$ part of the isometry group and a $U(1)$ from the $SO(3)$ part. Since the $SO(3)$ isometry descends from the $SU(2)$ action on $(u^i, -\bar{v}^i)$ writing

$$u^i = e^{i\theta + i\phi_3/2} w^i, \qquad v_i = e^{-i\theta + i\phi_3/2} z_i, \tag{4.12}$$

we see that θ parametrises the $U(1)$ action we mod out by and ϕ_3 parametrise the $U(1)$ subgroup of $SU(2)$. (Note that the $SU(2)$ element $\mathrm{diag}(-1,-1)$ is actually part of the $U(1)$ factor by which we mod out and hence the $SU(2)$ action descends to an $SO(3)$ action on $N(1,1)_I$.) The constraints $|u^i|^2 = |v_i|^2 = 1$ imply that w^i and z_i parametrise a pair of $\mathbb{C}P^2$ manifolds. In addition $u^i v_i = 0$ implies $w^i z_i = 0$. In terms of these constrained variables, reducing from the flat metric on $\mathbb{C}^6$ it is easy to show that the metric on $N(1,1)_I$ has the form

$$2\, ds^2 = d\bar{w}_i dw^i - |\bar{w}_i dw^i|^2 + d\bar{z}^i dz_i - |\bar{z}^i dz_i|^2 + \tfrac{1}{2}(d\phi_3 - i(\bar{w}_i dw^i + \bar{z}^i dz_i))^2, \tag{4.13}$$

The term $d\bar{w}_i dw^i - |\bar{w}_i dw^i|^2$ is the standard Fubini–Study metric of $\mathbb{C}P^2$ and similarly for z_i, though one must remember, of course that the coordinates are related by the constraint $w^i z_i = 0$. We can introduce an explicit parametrisation of w^i and z_i,

$$\begin{aligned}
(w^1, w^2, w^3) &= e^{-i(\alpha_1+\alpha_2)/3}(\mu_1 e^{i\alpha_1}, \mu_2 e^{i\alpha_2}, \mu_3)\,, \qquad \mu_1^2 + \mu_2^2 + \mu_3^2 = 1\,, \\
(z_1, z_2, z_3) &= e^{-i(\beta_1+\beta_2)/3}(\nu_1 e^{i\beta_1}, \nu_2 e^{i\beta_2}, \nu_3)\,, \qquad \nu_1^2 + \nu_2^2 + \nu_3^2 = 1\,.
\end{aligned} \tag{4.14}$$

The component parts of the metric can then be written as

$$
\begin{aligned}
d\bar{w}_i dw^i - |\bar{w}_i dw^i|^2 &= \sum_{i=1}^{3} d\mu_i^2 + \mu_1^2(1-\mu_1^2)d\alpha_1^2 + \mu_2^2(1-\mu_2^2)d\alpha_2^2 - 2\mu_1^2\mu_2^2 d\alpha_1 d\alpha_2 ,\\
-i\bar{w}_i dw^i &= (\mu_1^2 - \frac{1}{3})d\alpha_1 + (\mu_2^2 - \frac{1}{3})d\alpha_2 \,, \qquad (4.15)
\end{aligned}
$$

with similar expressions associated to z_i. The constraint $w^i z_i = 0$ then reads

$$
e^{i(\alpha_1+\beta_1)}\mu_1\nu_1 + e^{i(\alpha_2+\beta_2)}\mu_2\nu_2 + \mu_3\nu_3 = 0. \qquad (4.16)
$$

Finally we must identify the $U(1)^2$ subgroup of the $SU(3)$ isometries. Changing variables

$$
\alpha_r = \rho_r + \phi_r, \qquad \beta_r = \rho_r - \phi_r, \qquad r = 1,2\,, \qquad (4.17)
$$

it is easy to see that the $U(1)^2$ subgroup acts by shifting ϕ_r leaving ρ_r invariant. In addition the constraint $w^i z_i = 0$ now involves only ρ_r, namely

$$
e^{i2\rho_1}\mu_1\nu_1 + e^{i2\rho_2}\mu_2\nu_2 + \mu_3\nu_3 = 0. \qquad (4.18)
$$

In summary, ϕ_1, ϕ_2 and ϕ_3 parametrise the maximal $U(1)^3$ of the $SU(3)\times SU(2)$ isometry group which we use for the γ-deformation. Since there is no real three dimensional representation of $SU(3)$, the three Killing spinors on $N(1,1)_I$ must be invariant under the action of $SU(3)$ though transform as a triplet under the $SO(3)$ R-symmetry. Following the discussion in section 3.3, we thus conclude that this $U(1)^3$ action preserves a single Killing spinor.

Following the general discussions in the section 3, we can now give the data required to obtain the γ-deformed solution. We find

$$
\begin{aligned}
16\Delta &= \mu_1^2\mu_2^2\mu_3^2 + \nu_1^2\nu_2^2\nu_3^2 + \left[\mu_1^2\nu_1^2(\mu_2^2\nu_3^2 + \mu_3^2\nu_2^2) + (\text{cyclic})\right] \,,\\
\Delta^{1/3}M_{ab} &= \frac{1}{4}\begin{pmatrix} -(\mu_1^2+\nu_1^2)(\mu_1^2+\nu_1^2-2) & -(\mu_1^2+\nu_1^2)(\mu_2^2+\nu_2^2) & (\mu_1^2-\nu_1^2)\\ \cdot & -(\mu_2^2+\nu_2^2)(\mu_2^2+\nu_2^2-2) & (\mu_2^2-\nu_2^2)\\ \cdot & \cdot & 1 \end{pmatrix}\\
16\Delta A^1 &= \left[\mu_1^2\mu_2^2\mu_3^2 + \mu_1^2\nu_2^2(1-\mu_1^2-\nu_2^2+\mu_1^2\nu_2^2) - (\mu\leftrightarrow\nu)\right]d\rho_1\\
&\quad + 2\left[\mu_1^2\mu_2^2\nu_2^2(\nu_2^2-1) - (\mu\leftrightarrow\nu)\right]d\rho_2\,,\\
16\Delta A^2 &= \left[\mu_1^2\mu_2^2\mu_3^2 - \mu_1^2\nu_2^2(1-\mu_1^2-\nu_2^2+\mu_1^2\nu_2^2) - (\mu\leftrightarrow\nu)\right]d\rho_2\\
&\quad + 2\left[\mu_1^2\mu_2^2\nu_1^2(\nu_1^2-1) - (\mu\leftrightarrow\nu)\right]d\rho_1\,,\\
24\Delta A^3 &= \left[-\mu_1^2\mu_2^2\mu_3^2 - \mu_1^2\nu_2^2(1-\mu_1^2-\nu_2^2+\mu_1^2\nu_2^2) + 3\mu_1^2\mu_2^2\nu_1^2(1+\mu_3^2-\nu_1^2)\right.\\
&\quad \left. -2\mu_1^2\mu_2^2\nu_1^2\nu_2^2 + (\mu\leftrightarrow\nu)\right]d\rho_1 + [1\leftrightarrow 2]\,d\rho_2\,,\\
ds_4^2 &= \frac{1}{2}\sum_i^3 (d\mu_i^2 + d\nu_i^2) + ds^2_{\rho_1,\rho_2}\,,\\
-8\Delta\, ds^2_{\rho_1,\rho_2} &\equiv \left[\mu_1^2\mu_2^2\mu_3^2\nu_1^2(\nu_1^2-1) + (\mu\leftrightarrow\nu)\right]d\rho_1^2 + [1\leftrightarrow 2]\,d\rho_2^2\\
&\quad + 2\mu_1^2\mu_2^2\nu_1^2\nu_2^2(\mu_3^2+\nu_3^2)d\rho_1 d\rho_2\,. \qquad (4.19)
\end{aligned}
$$

In the discussion of the gauge theory dual, it will be useful to know the action of the Weyl group of $SU(3)$ on the deformation. In terms of elements of $SU(3)$ one is interested in the discrete subgroup generated by the maps

$$\begin{aligned} a_1 &: (u^1, u^2, u^3) \mapsto (u^2, -u^1, u^3), \\ a_2 &: (u^1, u^2, u^3) \mapsto (u^1, u^3, -u^2), \\ a_3 &: (u^1, u^2, u^3) \mapsto (-u^3, u^2, u^1), \end{aligned} \tag{4.20}$$

together with similar expressions for v_i. The Weyl group is then the quotient of this subgroup by those elements in Cartan subgroup $U(1)^2 \subset SU(3)$. The original metric is clearly invariant under this group since it is a subgroup of the isometry group $SU(3)$. The question is whether the deformed metric is also invariant. The action on our coordinates for the first element a_1 in (4.20) is straightforward: it permutes the pairs (μ_1, μ_2), (ν_1, ν_2), $(d\phi_1, d\phi_2)$ and $(d\rho_1, d\rho_2)$. The other elements are a little more complicated. For instance the second element a_2 translates into

$$\begin{pmatrix} \mu_1' \\ \mu_2' \end{pmatrix} = \begin{pmatrix} \mu_1 \\ (1-\mu_1^2-\mu_2^2)^{1/2} \end{pmatrix}, \quad \begin{pmatrix} d\phi_1' \\ d\phi_2' \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} d\phi_1 \\ d\phi_2 \end{pmatrix} \equiv S \begin{pmatrix} d\phi_1 \\ d\phi_2 \end{pmatrix}, \tag{4.21}$$

with similar expressions for ν_r and ρ_r. It is straightforward to see that this transformation leaves the deformed metric invariant whereas it changes the sign of the part of F_4 linear in γ. We return to this issue in section 5.

4.3 Weak G_2

We argued above that any weak G_2 manifold, with precisely one Killing spinor, admitting an isometric T^3 action will lead to a deformed solution preserving one supersymmetry. One example in this class is $N(1,1)_{II}$, the squashed Einstein metric on $N(1,1)$. It is straightforward to construct the deformed solution using the coordinates in (4.10), but as the expressions are a bit complicated we will not present them here. More general Aloff-Wallach spaces $N(k,l)$ have two weak G_2 metrics each with $SU(3) \times U(1)$ isometry and as this includes a T^3 action we can similarly construct the γ-deformed solutions.

The weak G_2 example that we will explicitly consider here is the squashed seven-sphere. This is given by the coset $Sp(2)\times Sp(1)/Sp(1)\times Sp(1)$ and has $SO(5)\times SO(3)$ isometry [32]. The squashing can be obtained by viewing the S^7 as an S^3 bundle over S^4 and changing the size of the S^3 fibres relative to the S^4 base. Apart from the round S^7 metric, this procedure leads to only one more Einstein metric given by

$$\frac{20}{9}\, ds^2_{squashed} \;=\; d\mu^2 + \frac{1}{4}\sin^2\mu\, \sigma_i^2 + \frac{1}{5}(\Sigma_i - \cos^2\frac{\mu}{2}\;\sigma_i)^2\,, \tag{4.22}$$

where the σ's and Σ's are left-invariant one forms in $SU(2)$. Here we take

$$\begin{aligned} \sigma_1 &= \cos\psi d\theta + \sin\psi \sin\theta d\phi\,, \\ \sigma_2 &= -\sin\psi d\theta + \cos\psi \sin\theta d\phi\,, \\ \sigma_3 &= d\psi + \cos\theta d\phi\,. \end{aligned} \tag{4.23}$$

The same expressions hold for the Σ's, putting 'primes' over the angles. Note also that the factor of 20/9 ensures that $R_{\mu\nu} = 6g_{\mu\nu}$.

In these co-ordinates, only an $SU(2)\times SU(2)\times U(1)$ isometry is manifest, where the $U(1)$ corresponds to simultaneous shifts of ψ and ψ'. In order to perform the deformation, we define $\psi_\pm = (\psi \pm \psi')/4$ and select the T^3 generated by the Killing vectors $(\partial_\phi, \partial_{\phi'}, \partial_{\psi_+})$. With the definitions given, the three angles on T^3 have the same period 2π.

As explained above, the deformation is completely specified by computing the following data:

$$
\begin{aligned}
\alpha^2\Delta &= -16c_\theta^2 s_\mu^4 - 16(3+2c_\mu)c_{\mu/2}^4(c_\theta c_{\theta'} + c_{4\psi_-}s_\theta s_{\theta'})^2 s_{\mu/2}^2 \\
&\quad -32c_\theta c_{\theta'}c_{\mu/2}^2(c_\theta c_{\theta'} + c_{4\psi_-}s_\theta s_{\theta'})s_\mu^2 s_{\mu/2}^2 + (4c_{\mu/2}^4 + 5s_\mu^2)(5s_\mu^2 + 4s_{\theta'}^2 s_{\mu/2}^4)\,,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\alpha^2}{2}\Delta A^1 &= 2s_\mu^2(s_{\theta'}(5c_\theta(c_{\theta'}c_{\mu/2}^2 s_{4\psi_-}d\theta + 4s_{\theta'}\,d\psi_-) + c_{4\psi_-}c_{\mu/2}^2 s_\theta(-20c_{\theta'}d\psi_- + (3+2c_\mu)s_{4\psi_-}s_{\theta'}\,d\theta)) \\
&\quad + s_{4\psi_-}s_\theta(-3-2c_\mu + c_{\theta'}^2 s_{\mu/2}^2)\,d\theta')\,, \\
\frac{\alpha^2}{2}\Delta A^2 &= 2c_{\mu/2}^2 s_\mu^2(-5c_{\theta'}s_\theta(c_\theta s_{4\psi_-}d\theta' + 4(3-2c_\mu)s_\theta\,d\psi_-) + s_{\theta'}(c_{4\psi_-}s_\theta(20c_\theta d\psi_- \\
&\quad -(3+2c_\mu)s_{4\psi_-}s_\theta\,d\theta') + s_{4\psi_-}(5+16c_{\mu/2}^2 s_\theta^2 s_{\mu/2}^2)\,d\theta))\,, \\
\frac{\alpha^2}{2}\Delta A^3 &= s_\mu^2(c_{\mu/2}^4(-s_{4\psi_-}s_{\theta'}(c_{\theta'} + 4c_\theta^2 c_{\theta'} + 2c_{4\psi_-}s_{2\theta}s_{\theta'})\,d\theta + s_\theta^2(4 + c_{\theta'}^2 - 4c_{4\psi_-}^2 s_{\theta'}^2)\,d\psi_-) \\
&\quad +5s_{\mu/2}^2(2+3c_{\theta'}^2 - (4+c_{\theta'}^2)s_{\mu/2}^2)\,d\psi_- + c_{\mu/2}^2(s_{4\psi_-}s_\theta(c_\theta(4+c_{\theta'}^2) + c_{4\psi_-}c_{\theta'}s_\theta s_{\theta'})d\theta' \\
&\quad -5c_{\theta'}s_{4\psi_-}s_{\theta'}s_{\mu/2}^2\,d\theta + (2 - 12c_\theta^2 + 3c_{\theta'}^2 + 7c_\theta^2 c_{\theta'}^2 - 2c_{4\psi_-}^2 c_\mu s_\theta^2 s_{\theta'}^2 \\
&\quad -(-4+9c_\theta^2)(4+c_{\theta'}^2)s_{\mu/2}^2)\,d\psi_-))\,,
\end{aligned}
$$

$$
\beta^2 M_{ab} = \Delta^{-1/3}\begin{pmatrix} 4c_{\mu/2}^4 + 5s_\mu^2 & -4c_{\mu/2}^2(c_\theta c_{\theta'} + c_{4\psi_-}s_\theta s_{\theta'}) & 2c_\theta(-4c_{\mu/2}^2 + 4c_{\mu/2}^4 + 5s_\mu^2) \\ \cdot & 4 & 8c_{\theta'}s_{\mu/2}^2 \\ \cdot & \cdot & 4(5s_\mu^2 + 4s_{\mu/2}^4) \end{pmatrix}\,,
$$

where $\alpha \equiv 2000/27$ and $\beta \equiv 20/3$. The expression for ds_4^2 is a bit long, so we prefer to leave it as implicitly defined by

$$
ds_4^2 = ds_{squashed}^2 - \Delta^{1/3}M_{ab}(d\varphi^a + A^a)(d\varphi^b + A^b)\,. \tag{4.24}
$$

Again, after the deformation, the resulting configuration just amounts to substituting these values in (3.15)-(3.16).

5 Deformations of the dual field theory

The γ-deformed supergravity solutions we studied in the previous section should correspond to an exactly marginal deformation of the dual $d=3$ conformal field theory. We will now identify the relevant operators for the cases in which a proposal for the dual field theory has been made.

Lunin and Maldacena showed that for the $d=4$ CFTs with AdS_5 duals, the γ-deformation results in a sort of star-product amongst the elementary fields. This has the simple effect of introducing various γ-dependent phases into the Lagrangian. This

result was deduced using open string field theory and the fact that the γ-deformation is a result of a T-duality, a shift and another T-duality. This argument cannot be directly applied to the $d = 3$ CFTs studied here, because there is no analog of deriving the CFTs from open strings.

However, for small values of γ, there is an alternative way to find the exactly marginal operator. At the linearised level, the deformation in supergravity amounts to turning on a massless ($\Delta = d$) mode in the KK spectrum of the original background. The AdS/CFT dictionary then gives the exactly marginal operator in the CFT dual, which is the corresponding change in the dual field theory Lagrangian.

Consider the deformations of $\mathcal{N} = 4$ SYM studied in [3], for example, which preserve $\mathcal{N} = 1$ supersymmetry. The deformation in the field theory must correspond to perturbing the superpotential. The only term in the deformed supergravity solution that is linear in γ appears in the combination $B_{NS} + iB_{RR}$, which translates into an operator obtained by acting $Q_\alpha Q^\alpha$ on the chiral primary operator $\mathrm{Tr}(\Phi^1\Phi^2\Phi^3 + \Phi^3\Phi^2\Phi^1)$. Since the superpotentials enters the Lagrangian in the form $(\int d^2\theta W + c.c.)$, we conclude that the γ-deformation turns on the superpotential $W = \mathrm{Tr}(\Phi^1\Phi^2\Phi^3 + \Phi^3\Phi^2\Phi^1)$. In fact an easier way to obtain this superpotential is to observe that it is the unique $\Delta = 3$ chiral primary operator which breaks the $SU(3)$ global symmetry to $U(1)^2$. For small γ it indeed agrees with the general result of [3]: $W = \mathrm{Tr}(e^{i\pi\gamma}\Phi^1\Phi^2\Phi^3 - e^{-i\pi\gamma}\Phi^1\Phi^3\Phi^2)$.

Essentially the same story holds for all the examples considered in [3] and those that we consider here. Starting with a theory preserving 4 supercharges, the γ-deformation amounts to adding a superpotential to the field theory Lagrangian which (a) has $\Delta = d - 1$, (b) belongs to a short-multiplet in the superconformal algebra, and (c) breaks the global symmetry group into $U(1)^2$, for $\mathrm{AdS}_5 \times M_5$ examples, or $U(1)^3$ for $\mathrm{AdS}_4 \times X_7$ examples. This observation provides a quick way to identify the superpotential even without looking at the detailed form of the γ-deformed supergravity solution. For example, in the $T^{1,1}$ theory, $W = \mathrm{Tr}(A_+B_+A_-B_- + A_+B_-A_-B_+)$ is the unique $\Delta = 3$ chiral primary operator which breaks the $SU(2)^2$ global symmetry to $U(1)^2$, in agreement, for small γ, with the general result of [3]: $W = \mathrm{Tr}(e^{i\pi\gamma}A_+B_+A_-B_- - e^{-i\pi\gamma}A_+B_-A_-B_+)$.

Let us now turn to the $d = 3$ examples for which the dual CFT is known, as reviewed in section 2.

Q(1,1,1) and M(3,2) cases

The deformed solutions for $Q(1,1,1)$ and $M(3,2)$ both preserve $\mathcal{N} = 2$ supersymmetry, so we seek a superpotential that is chiral primary with dimension two and preserves $U(1)^2$. For the $Q(1,1,1)$ theory, we note that the chiral operators $\mathrm{Tr}(ABC)^2$, totally symmetrised over the $SU(2)$ indices, have $\Delta = 2$ and transform in the $(\mathbf{3},\mathbf{3},\mathbf{3})$ of $SU(2)^3$. This contains a single operator that is neutral under $U(1)^3$ and hence this must be the superpotential giving the γ-deformation.

A similar story holds for the $M(3,2)$ theory. The operator $\mathrm{Tr}(U^3V^2)$ in the totally symmetrized product $(\mathbf{10},\mathbf{3})$ of $SU(3) \times SU(2)$, is chiral with $\Delta = 2$. This again contains a unique state neutral under $U(1)^3$ and hence this must be the superpotential

giving the γ-deformation.

N(1,1)$_I$ case

Our last example is the $N(1,1)_I$ theory. From our discussion in section 4 we know that the γ-deformation breaks $\mathcal{N}=3$ to $\mathcal{N}=1$ and $SU(3)\times SO(3)_R$ to $U(1)^3$. Note that despite not having any surviving R-symmetry, there is a $U(1)\subset SO(3)_R$ which still acts on the matter fields and leads to a global symmetry of the action.

This breaking implies that the deformation of the SCFT must be by a term which preserves only $\mathcal{N}=1$ supersymmetry. As we reviewed in subsection 2.1, the theory contains three $\mathcal{N}=3$ hypermultiplets which in $\mathcal{N}=2$ language correspond to two chiral superfields u^i and v_i which can be naturally grouped as $U^i_\alpha=(u^i,-\bar{v}^i)$ and the conjugate $V_{i\alpha}=(v_i,\bar{u}_i)$. We can parametrise $\mathcal{N}=3$ superspace by supercoordinates $\theta^\pm,\theta^0$, which under $U(1)\subset SO(3)$ carry charge $\pm1,0$ respectively. The chiral fields u^i and $\bar{v}^i$ are then expansions in terms of θ^+ and θ^- respectively. However, we can also represent the hypermultiplets in terms of $\mathcal{N}=1$ multiplets, that is, as superfields which are functions of the $\mathcal{N}=1$ superspace with supercoordinate θ^0. In both descriptions, we can use the same name for a superfield and its lowest scalar component. However, since we are keeping a different subset of supercharges from $\mathcal{N}=3$, the scalars are combined with different components of fermions to form the superfields.

In section 2.1 we described the chiral primary operators in the conformal field theory using an $\mathcal{N}=2$ language. In particular, the most general chiral primary operator of dimension two is of the form

$$T^{kl}_{ij}\ C^{\alpha\beta\gamma\delta}\ \mathrm{Tr}\ U^i_\alpha\, V_{j\beta}\, U^k_\gamma\, V_{l\delta}\,, \tag{5.1}$$

where $C^{\alpha\beta\gamma\delta}$ and T^{kl}_{ij} are irreducible tensors in the $(\mathbf{27},\mathbf{5})$ of $SU(3)\times SU(2)_R$. Given the comments of the previous paragraph, we now view this operator as being written in terms of $\mathcal{N}=1$ superfields U and V. If we now integrate over the full $\mathcal{N}=1$ superspace we get a marginal deformation that breaks $\mathcal{N}=3$ to $\mathcal{N}=1$. Next, we need to recall that the γ-deformed solution is invariant under $U(1)\subset SO(3)_R$. But only one element in the $\mathbf{5}$ representation has this property: this implies that the only non-zero component of C, up to symmetrization, is C^{1122}. Finally, invariance under $U(1)^2\subset SU(3)$ still leaves three possibilities, as there are three states in the $\mathbf{27}$ of $SU(3)$ invariant under the Cartan subgroup.

In order to decide which of these three operators is turned on in the γ-deformed supergravity solution (4.19)–(3.16), we consider the transformation properties under the discrete subgroup generated by the elements (4.20). Acting on the $\mathbf{27}$ representation, this subgroup transforms the three $U(1)^2$-invariant states into each other. In fact, it gives a realization of the Weyl group of $SU(3)$. Furthermore, there is a unique combination of states which is invariant under all elements of the Weyl group. Expanding the γ-deformed solution in the parameter γ, to leading order, the metric is unchanged, but there is a deformation of the flux F_4 as given in (3.16) which, according to the Kaluza–Klein analysis, should transform as the $\mathbf{27}$ of $SU(3)$. Under the cyclic subgroup of the Weyl group generated by the element $b=a_2a_1$ it is easy

to see that the flux deformation is invariant. Thus it would appear that the deformation corresponds to the unique Weyl-singlet combination of operators. To complete this analysis one must show that the flux is also invariant under a_i separately. As stands, it appears that under these elements the linear term in F_4 maps to minus itself. Since there are no elements in the **27** representation with such properties this raises a puzzle.

6 Discussion

We have used the techniques of [3] to construct a large number of new supersymmetric solutions of $D = 11$ supergravity with AdS_4 factors. They correspond to exactly marginal deformations in the dual CFT. For certain examples, we managed to identify the operators in the CFT giving rise to the deformation for small values of the deformation parameter γ.

It would be interesting to explore this further. In particular it would be nice to be able to identify the exact deformation in the field theory for all values of γ as was done in the string theory setting in [3]. It would also be interesting to know if the field theories considered here have additional exactly marginal deformations and if so whether they too can be constructed in explicit form.

We have focused on deformations of supersymmetric AdS_4 solutions that preserve some supersymmetry. We now briefly discuss how some simple generalisations lead to new AdS_4 solutions with no supersymmetry. Of course the stability of such solutions, and hence their relevance to the AdS/CFT correspondence, now needs to be checked.

One possibility is to start with a supersymmetric solution and choose a T^3 action that doesn't preserve any supersymmetry. There are several possibilities of this type for the solutions considered in this paper. In addition one could also consider the supersymmetric solution based on $V_{5,2}$ which has a T^3 action but not one commuting with supersymmetry.

Another possibility is to deform a non-supersymmetric AdS_4 solution with a T^3 action. One case is to start with the supersymmetric solutions of this paper and simply reverse the sign of the four-form (or equivalently, reverse the orientation on X_7). As is well known this gives solutions that do not preserve any supersymmetry [29] but are nevertheless stable [34]. The γ-deformed solution has metric as in (3.15) and the four-form as in (3.16) but with the sign of the first two terms on the right hand side changed.

A second case is to recall that some of the supersymmetric solutions considered in this paper are members of more general families of homogeneous Einstein spaces. For example, $M(3,2)$ is a member of the $M(m,n)$ family which are all $U(1)$ bundles over $\mathbb{C}P^2 \times S^2$ (this is the same family as the $M^{pqr} \sim SU(3) \times SU(2) \times U(1)/SU(2) \times U(1) \times U(1)$ spaces introduced in [18]). Similarly, $Q(1,1,1)$ is a member of the $Q(p,q,r)$ family which are all $U(1)$ bundles over $S^2 \times S^2 \times S^2$ (this is the same family as the $Q^{p,q,r} \sim SU(2)^3 \times U(1)/U(1)^3$ introduced in [20]). These give rise to stable non-supersymmetric solutions as discussed in [35] and [36] for the $M(m,n)$ and $Q(p,q,r)$ families, respectively. The γ-deformed versions of these can easily be found.

A third case to consider is the non-supersymmetric "Englert " solutions [37, 38, 32]. These are obtained from a supersymmetric solution by switching on a four-form flux constructed as a bi-linear of a Killing spinor and flipping the orientation. It is known that they do not preserve any supersymmetry [39, 40]. Furthermore, they have been shown to be unstable if the supersymmetric solution preserves two or more supersymmetries [41]. In addition the Englert solution using the squashed S^7 was demonstrated to be unstable in [42], while some of the $N(k, l)$ have shown to be unstable [41, 43]. If we assume that there are some stable $N(k, l)$ Englert solutions then we can generate additional stable solutions. We need to check that the four-form is invariant under the T^3 action (in order to be able to reduce the solution to $D = 8$ and then employ the generating procedure), but this is guaranteed by its construction as a Killing spinor bilinear and, as noted at the beginning of section 4, the fact that the Killing spinor is invariant under the T^3 action.

A final possibility, again with non-vanishing four-form, are the solutions based on $U(1)$ bundles over $D = 6$ Kähler spaces [44, 45]. We are unaware of any results concerning the stability of this class of solutions. Nevertheless, some of these solutions admit a T^3 action and we can generate the corresponding γ-deformed solutions.

Note added: The results of this paper were presented by one of us (JPG) at the meeting *Workshop on Gravitational Aspects of String Theory* May 2-6, 2005 at Fields Institute, Toronto. Following this, but as we were finalising this preprint, the preprint [46] appeared which has some overlap with our results.

Appendix

A Baryons from $N(1,1)_I$

Baryonic states in field theory are dual to five-branes wrapping five-cycles in the manifold X_7. For supersymmetric states the five cycles must lift to supersymmetric six-cycles in the eight-dimensional cone $C(X_7)$ given in (2.1). In the case of $N(1,1)_I$, the cone is hyper-Kähler and so has $Sp(2)$ holonomy. A supersymmetric six-cycle on $C(N(1,1)_I)$ can at most be calibrated by the Hodge dual of one of the three Kähler forms. Equivalently, it will be holomorphic (a divisor) with respect to one of the three complex structures. As such it breaks one of the three supersymmetries on $C(N(1,1)_I)$. Consequently, in looking for supersymmetric cycles it is natural to concentrate on only one of the complex structures on the cone. Equivalently, one isolates one of the three Sasaki structures on $N(1,1)_I$.

Viewed as a regular Sasakian manifold, $N(1,1)_I$ can be written canonically as an explicit $U(1)$ fibration over a Kähler–Einstein (KE) six-manifold:

$$ds^2 = (d\psi' + \sigma) + ds_6^2 \,. \tag{A.1}$$

In fact ds_6^2 is the KE metric on the flag manifold $SU(3)/U(1) \times U(1)$, normalised so that $Ricci_6 = 8g_6$ and $d\sigma = 2J_6$. To make this structure explicit, we start with the

metric in the standard form of an $SO(3)$ fibration over $\mathbb{C}P^2$, as given in [22] (see also eqn. (4.10)),

$$\begin{aligned} 2ds^2 &= d\mu^2 + \tfrac{1}{4}\sin^2\mu(\sigma_1^2+\sigma_2^2) + \tfrac{1}{4}\sin^2\mu\cos^2\mu\,\sigma_3^2 \\ &+ \tfrac{1}{2}\left\{(\Sigma_1 - \cos\mu\sigma_1)^2 + (\Sigma_2 - \cos\mu\sigma_2)^2 + (\Sigma_3 - \tfrac{1}{2}(1+\cos^2\mu)\sigma_3)^2\right\}\,, \end{aligned} \tag{A.2}$$

where Σ_i are right-invariant one-forms on $SO(3)$, and σ_i are right-invariant one-forms on $SU(2)$. Let us introduce coordinates so that

$$\begin{aligned} \Sigma_1 &= \sin\phi d\theta - \cos\phi\sin\theta d\psi \\ \Sigma_2 &= \cos\phi d\theta + \sin\phi\sin\theta d\psi \\ \Sigma_3 &= d\phi + \cos\theta d\psi\,, \end{aligned} \tag{A.3}$$

with $0 < \theta < \pi$, $0 < \phi < 2\pi$ and $0 < \psi < 2\pi$ so that we parametrise $SO(3)$ rather than $SU(2)$. Note that we have scaled the seven-dimensional metric so that $Ricci = 6g$.

By defining $\psi' = -\psi/2$ and completing the square, the metric can be written as in (A.1) above with the 6-metric on the flag manifold given by

$$\begin{aligned} ds_6^2 &= \tfrac{1}{4}\left[d\theta - \cos\mu(\sin\phi\sigma_1 + \cos\phi\sigma_2)\right]^2 \\ &+ \tfrac{1}{4}\sin^2\theta\left[d\phi - \cos\mu\cot\theta(\cos\phi\sigma_1 - \sin\phi\sigma_2) - \tfrac{1}{2}(1+\cos^2\mu)\sigma_3\right]^2 \\ &+ \tfrac{1}{2}\left[d\mu^2 + \tfrac{1}{4}\sin^2\mu(\sigma_1^2+\sigma_2^2) + \tfrac{1}{4}\sin^2\mu\cos^2\mu\sigma_3^2\right]\,, \end{aligned} \tag{A.4}$$

which displays it explicitly as an S^2 bundle over $\mathbb{C}P^2$. In addition we have the one-form σ in (A.1) given by

$$\sigma = -\tfrac{1}{2}\cos\mu\sin\theta(\cos\phi\sigma_1 - \sin\phi\sigma_2) + \tfrac{1}{4}\cos\theta(1+\cos^2\mu)\sigma_3 - \tfrac{1}{2}\cos\theta d\phi\,, \tag{A.5}$$

which indeed satisfies $d\sigma = 2J_6$ where

$$\begin{aligned} J_6 &= \tfrac{1}{4}\sin\theta\left[d\theta - \cos\mu(\sin\phi\sigma_1 + \cos\phi\sigma_2)\right] \\ &\quad \wedge\left[d\phi - \cos\mu\cot\theta(\cos\phi\sigma_1 - \sin\phi\sigma_2) - \tfrac{1}{2}(1+\cos^2\mu)\sigma_3\right] \\ &+ \cos\phi\sin\theta K^1 - \sin\phi\sin\theta K^2 - \cos\theta K^3 \end{aligned} \tag{A.6}$$

and K^i are the following basis for self-dual two-forms on $\mathbb{C}P^2$:

$$\begin{aligned} K^1 &= \tfrac{1}{4}\sin\mu d\mu\wedge\sigma_1 + \tfrac{1}{8}\cos\mu\sin^2\mu\sigma_2\wedge\sigma_3 \\ K^2 &= \tfrac{1}{4}\sin\mu d\mu\wedge\sigma_2 + \tfrac{1}{8}\cos\mu\sin^2\mu\sigma_3\wedge\sigma_1 \\ K^3 &= \tfrac{1}{4}\cos\mu\sin\mu d\mu\wedge\sigma_3 + \tfrac{1}{8}\sin^2\mu\sigma_1\wedge\sigma_2\,. \end{aligned} \tag{A.7}$$

We now consider the 5-cycles Σ_5 on the seven-manifold defined by either $\theta = 0$ or $\theta = \pi$. These are $U(1)$ bundles over $\mathbb{C}P^2$. They are supersymmetric cycles provided that the corresponding 6-cycles on the Calabi-Yau four-fold cone with base $N(1,1)_I$ are in fact divisors. In our normalisations, the metric on the Calabi-Yau cone is given by

$$ds_8^2 = dr^2 + r^2[(d\psi' + \sigma)^2 + ds_6^2]\,, \tag{A.8}$$

with the corresponding Kähler-form given by

$$J_8 = r dr \wedge (d\psi' + \sigma) + r^2 J_6 \,. \tag{A.9}$$

It can now easily be checked that the relevant six-cycles are indeed calibrated by $\frac{1}{6}(J_8)^3$. M5-branes wrapped on these cycles are expected to correspond to baryons in the dual SCFT. The conformal dimension of these baryon operators will be given by the geometric formula [47, 12]

$$\Delta = \frac{\pi N}{6} \frac{\mathrm{Vol}(\Sigma_5)}{\mathrm{Vol}(X_7)} \,. \tag{A.10}$$

In calculating the volume of the five-cycle with, say, $\theta = 0$, one first needs to shift the coordinate $\psi \to \psi - \phi$ to ensure that the coordinates are well defined at $\theta = 0$. Having done this we find $\Delta = N/2$ which exactly agrees with that expected from the description of the field theory dual to $N(1,1)_I$ presented in [13]. In particular the operators $\det(U^i_\alpha)$ have conformal dimension $N/2$.

It would be interesting to determine the stability groups of the supersymmetric cycles and then quantise the collective coordinates of the wrapped five-brane, following the calculations in [48, 12]. This would give predictions for the representations under the global symmetry group $SU(3) \times SO(3)$ that the baryons carry and this could be compared with the representations in $\det(U^i_\alpha)$ that are chiral primary.

B Parametrisations of $N(1,1)$

We have used two different sets of coordinates on $N(1,1)$. One, following [22], is the standard set where the manifold is explicitly an $SO(3)$ fibration over $\mathbb{C}P^2$. The $N(1,1)_I$ and $N(1,1)_{II}$ metrics in these coordinates are given in eqn. (4.10). The other set, discussed in section 4.2, was adapted to the construction of $N(1,1)_I$ as a hyper-Kähler quotient. In this appendix we show how these two parametrisations are related. The easiest way to do this is to recall that topologically $N(1,1) \simeq SU(3)/U(1)$. The two coordinate systems are then related by two different parametrisations of $SU(3)$.

Specifically $N(1,1)$ is defined to be the coset $SU(3)/U(1)$ where the $U(1)$ is embedded in the $\lambda_8 = \frac{1}{\sqrt{3}}\mathrm{diag}(1,1,-2)$ direction. Let us write a general $SU(3)$ element as

$$g = \begin{pmatrix} u^1 & -\bar{v}^1 & t^1 \\ u^2 & -\bar{v}^2 & t^2 \\ u^3 & -\bar{v}^3 & t^3 \end{pmatrix} \in SU(3) \,. \tag{B.1}$$

For (u^i, v_i), the $SU(3)$ condition implies that $|u^i|^2 = |v^i|^2 = 1$ and $u^i v_i = 0$. Identifying (u^i, v_i) with the corresponding variables in section 4.2, we see that these conditions exactly match the hyper-Kähler quotient conditions together with fixing the radius on the cone $C(N(1,1)_I)$. Once a value of (u^i, v_i) satisfying these constraints is chosen, the other $SU(3)$ constraints involving t^i can be solved completely by setting $\bar{t}_i = -\epsilon_{ijk} u^j \bar{v}^k$. So, in the discussion below, we regard t^i as a

function of (u^i, v^i). In the hyper-Kähler quotient we also modded out by the $U(1)$ action $(u^i, v^i) \sim (e^{i\theta} u^i, e^{-i\theta} v^i)$. For consistency this must correspond to the action of $\lambda_8 = \frac{1}{\sqrt{3}} \mathrm{diag}(1, 1, -2)$ on $g \in SU(3)$. It is easy to see that this indeed that case.

The parametrization [22] of $N(1,1)$ as an $SO(3)$ fibration over $\mathbb{C}P^2$ is related to the well-known identification $\mathbb{C}P^2 \simeq (SU(3)/\mathbb{Z}_2)/U(2)$. To form this coset we write the general $SU(3)$ element (following the discussion in appendix A of [21]) as

$$g = \begin{pmatrix} \bar{a}_1 & \bar{a}_2 & 0 \\ -a^2 & a^1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\mu & 0 & \sin\mu \\ 0 & 1 & 0 \\ -\sin\mu & 0 & \cos\mu \end{pmatrix} \begin{pmatrix} b^1 & -\bar{b}_2 & 0 \\ b^2 & \bar{b}_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-2i\theta} \end{pmatrix}. \tag{B.2}$$

where a^α and b^α form unit quaternions. The last factor is the $U(1)$ subgroup used to form $N(1,1) \simeq SU(3)/U(1)$. Together with the second last factor it forms an $U(2)$ subgroup. Note that the unit quaternion made of b^α actually parametrises $S^3/\mathbb{Z}_2 \simeq SO(3)$ since $(b^1, b^2) \sim (-b^1, -b^2)$ are identified by the $U(1)$ element with $\theta = \pi$. The other unit quaternion a^α and the real number μ are coordinates on the $\mathbb{C}P^2$ base. Thus quotienting by $U(1)$ we see explicitly that $N(1,1)$ is a $SO(3)$ fibration over $\mathbb{C}P^2$.

Comparing (B.2) to the parametrization (B.1) we get an explicit transformation between the coordinates on $N(1,1)$ used in section 4.2 and those of ref. [22]. In particular, it is easy to show, writing σ_i and Σ_i for the right-invariant one-forms constructed from a^α and b^α, respectively, that the two forms of the $N(1,1)_I$ metric (4.10) and (4.13) are equivalent.

Let us end by noting that the $N(1,1)_I$ metric is *not* the same as the usual coset metric obtained by simply gauging away the $U(1)$ factor from the standard $ds^2 = -\mathrm{Tr}(g^{-1}dg)^2$ of $SU(3)$. The latter in the notation of section 4.2 is given by

$$\begin{aligned} ds^2 &= d\bar{w}_i dw^i - |\bar{w}_i dw^i|^2 + d\bar{z}^i dz_i - |\bar{z}^i dz_i|^2 + d\bar{s}_i ds^i - |\bar{s}_i ds^i|^2 \\ &\quad + \tfrac{1}{2}(d\phi^3 + i(\bar{w}_i dw^i + \bar{z}^i dz_i))^2 \end{aligned} \tag{B.3}$$

where w^i and z_i are coordinates on $\mathbb{C}P^2$ satisfying $w^i z_i = 0$ while $\bar{s}_i = -\epsilon_{ijk} w^j \bar{z}^k$. This differs from (4.13) by an extra $\mathbb{C}P^2$-metric factor for s^i.

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